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Transformed Rational Chebyshev Approximation

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Let X be a compact Hausdorff space and $C(X)$ be the space of continuous real functions on X. For h a continuous function from X into the extended real line, define

$$\| h \| = \sup \{ | h(x) | : x \in X \}.$$

Let $\{\phi_1, ..., \phi_n\}$, $\{\psi_1, ..., \psi_m\}$ be linearly independent subsets of $C(X)$ and define

$$R(A, x) = P(A, x)/Q(A, x) = \sum_{k=1}^{n} a_k \phi_k(x) \Big/ \sum_{k=1}^{m} a_{n+k} \psi_k(x).$$

Let $\mathscr{P} = \{A : Q(A, \cdot) > 0\}$. Let w be a continuous mapping of the Cartesian product of X and the real line into the extended real line. Approximations are of the form

$$F(A, x) = w(x, R(A, x)) \qquad A \in \mathscr{P} \subset E_{n+m}.$$

The approximation problem is: Given $f \in C(X)$, to find a coefficient vector A^* minimizing $e(A) = \| f - F(A, \cdot) \|$ over $A \in \mathscr{P}$. Such a coefficient vector A^* is called best and $F(A^*, \cdot)$ is called a best approximation to f on X.

w is a transformation operator. The first study of transformations was that of the author [2], who studied transformations of ordinary rational functions on an interval. Kaufman and Belford [6] have studied transformations of alternating families. Williams [9] has studied some special cases of transformations of Haar subspaces on an interval.

PRELIMINARIES

We will call w a *weak ordering function* if for all $x \in X$, either

(i) $w(x, \cdot)$ is constant, or

(ii) $w(x, \cdot)$ is monotonic and strictly monotonic where it is finite.

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If (i) does not occur, w is called an *ordering function*.

Ordering functions of near full generality were first considered by Kaufman and Belford. Let σ be a continuous mapping of the real line into the extended real line which is monotonic and strictly monotonic where it is finite. Let r be an element of $C(X)$. If r has no zeros, $w(x, y) = r(x)\,\sigma(y)$ is an ordering function. If σ does not take infinite values, $w(x, y) = r(x)\,\sigma(y)$ is a weak ordering function. Special cases where w is of this form have been considered by the author [2, 5] and Williams [9]. We assume henceforth (unless stated otherwise) that w is a weak ordering function.

Allowing (i) to happen may seem to the reader to be of no practical utility. However, the approach of Williams [9] to curve fitting does involve transformations which often map into zero at some points x of X.

To avoid trivial cases, we assume henceforth that

$$\mathscr{P}' = \{A : A \in \mathscr{P}, \| F(A, \cdot)\| < \infty\}$$

is nonempty, which implies that unbounded approximations cannot be best.

Part of the analysis in this paper is in terms of the betweeness property, introduced by the author in [3].

DEFINITION. A subset G of $C(X)$ has the *betweeness property* if for given $g_0, g_1 \in G$, there is a λ-set $\{h_\lambda\} \in G$ such that $h_0 = g_0$, $h_1 = g_1$, and for all $x \in X$, h_λ is either a strictly monotonic continuous function of λ or a constant, $0 \leqslant \lambda \leqslant 1$.

The family $\{R(A, \cdot) : A \in \mathscr{P}\}$ has the betweenness property [3, p. 152].

LEMMA 1. *Let G be a subset of $C(X)$ with the betweenness property. Let w be a weak ordering function. Let*

$$w(G) = \{h : h = w(\cdot, g), g \in G, \| h \| < \infty\}.$$

Then $w(G)$ has the betweenness property.

CHARACTERIZATION OF BEST APPROXIMATIONS

Let $M(A) = \{x : | f(x) - F(A, x)| = e(A)\}$. By continuity of $| f - F(A, \cdot)|$ into the extended real line and compactness of X, $M(A)$ is a nonempty set.

THEOREM 1. *A necessary and sufficient condition for A to be best, where $0 < e(A) < \infty$, is that no $B \in \mathscr{P}'$ exist with*

$$(F(B, x) - F(A, x))\,(f(x) - F(A, x)) > 0 \qquad x \in M(A).$$

This follows directly from the corollary to Theorem 1 of [3].

To get more convenient results, we need to consider the direction of monotonicity of w. Define $s(x) = 0$ if $w(x, \cdot)$ is constant, $s(x) = 1$ if $w(x, \cdot)$ is monotonic increasing, and $s(x) = -1$ if $w(x, \cdot)$ is monotonic decreasing.

COROLLARY. *A necessary and sufficient condition that A be best, where $0 < e(A) < \infty$, is that no $B \in \mathscr{P}'$ exist with*

$$s(x)\,[P(B, x)\,Q(A, x) - P(A, x)\,Q(B, x)][f(x) - F(A, x)] > 0, \qquad x \in M(A). \tag{1}$$

Proof. For $B \in \mathscr{P}'$,

$$\begin{aligned} &\operatorname{sgn}(F(B, x) - F(A, x)) \\ &\quad = s(x)\operatorname{sgn}(R(B, x) - R(A, x)) \\ &\quad = s(x)\operatorname{sgn}([P(B, x)\,Q(A, x) - P(A, x)\,Q(B, x)]/[Q(A, x)\,Q(B, x)] \\ &\quad = s(x)\operatorname{sgn}[P(B, x)\,Q(A, x) - P(A, x)\,Q(B, x)]. \end{aligned}$$

COROLLARY. *A necessary and sufficient condition that A be best, where $0 < e(A) < \infty$, is that no B exist with* (1) *holding.*

Proof. If such B does not exist, apply the sufficiency part of the previous corollary. If such B does exist, let $C_\lambda = A + \lambda B$. For all $\lambda > 0$ and sufficiently small, $C_\lambda \in \mathscr{P}'$. (1) is satisfied with $B = C_\lambda$. Apply the necessity part of the previous corollary.

Associated with the parameter A we have the linear space

$$S(A) = \{P(A, \cdot)\,Q(B, \cdot) - Q(A, \cdot)\,P(B, \cdot) : B \in E_{n+m}\},$$

of dimension at most $n + m - 1$ [1, p. 159].

Let $\{\theta_1, \ldots, \theta_p\}$ be a basis of $S(A)$ and

$$\Phi(x) = (\theta_1(x), \ldots, \theta_p(x)).$$

By the theorem on linear inequalities [1, p. 19], we have

COROLLARY. *A necessary and sufficient condition that A be best, where $0 < e(A) < \infty$, is that* 0 *is in the convex hull of*

$$\{(f(x) - F(A, x))\,s(x)\,\Phi(x) : x \in M(A)\}.$$

CONVEXITY OF THE SET OF BEST PARAMETERS

Let $\mathscr{A}^*$ be the set of best parameters. In the following lemma we do not assume that w is a weak ordering function.

LEMMA 2. *$\mathscr{A}^*$ is convex if $w(x, \cdot)$ is monotone for all $x \in X$.*

Proof. Let $A, B \in \mathcal{C}^*$. Consider the parameter $C = \lambda A + (1 - \lambda) B$ for given λ in $(0, 1)$. We have for given $x \in X$, $R(C, x)$ being between $R(A, x)$ and $R(B, x)$ [3, p. 152]. Hence $F(C, x)$ is between $F(A, x)$ and $F(B, x)$ and $f(x) - F(C, x)$ is between $f(x) - F(A, x)$ and $f(x) - F(B, x)$. Thus $e(C) \leqslant \max \{e(A), e(B)\}$.

Sets on Which all Best Approximations Agree

The following terminology is due to Lawson [7, pp. 22–23].

Definition. A subset Y of X is an error-determining set (ED set) for f if

$$\inf\{\sup\{|f(x) - F(A, x)| : x \in X\} : A \in \mathcal{P}\}$$
$$= \inf\{\sup\{|f(x) - F(A, x)| : x \in Y\} : A \in \mathcal{P}\}.$$

An *irreducible error-determining set* (IED set) for f is an ED set for f which has no proper subset which is an ED set for f.

Lemma 3. *An* IED *set for f always exists, contains at most $n + m$ points, and is a subset of $M(A)$ for all A best.*

The above lemma is a consequence of the last corollary to Theorem 1 and the theorem of Caratheodory [1, p. 17].

Lemma 4. *Best approximations to f agree on any* IED *set for f.*

Proof. Let A, B be best to f and let Y be a set on which $F(A, \cdot)$ and $F(B, \cdot)$ differ, say at the point x. By convexity of the set of best coefficients, $(A + B)/2$ is also best. Further $x \notin M((A + B)/2)$. Hence x cannot be in an IED set for f.

Uniqueness

Definition. x is a *fixed point* of F if all approximants take the same value at x. Let V be the set of fixed points of F. By continuity of w, V is closed. Let $C(V, X)$ denote the set of continuous functions taking the same values on V.

Theorem 2. *Let $F(A, \cdot)$ be best to f. Let $S(A)$ be of dimension l on $X \sim V$ and a Haar subspace of dimension l on an* IED *set Y for f. Let s not vanish on Y. Then Y has exactly $l + 1$ points and $F(A, \cdot)$ is uniquely best to f.*

The proof is similar to that of Theorem 3 of [5], with the corollaries to Theorem 1 being used.

COROLLARY. *Let $F(A, \cdot)$ be best to f. Let $S(A)$ be of dimension l on $X \sim V$ and a Haar subspace of dimension l on $M(A)$. Let s not vanish on $M(A)$. Then $F(A, \cdot)$ is uniquely best to f.*

THEOREM 3. *A necessary and sufficient condition that $F(A, \cdot)$ be uniquely best when it is best to $f \in C(V, X)$ is that $S(A)$ is a Haar subspace on $X \sim V$.*

Proof. Let l be the dimension of $S(A)$ on $X \sim V$. *Sufficiency.* If $f = F(A, \cdot)$, $F(A, \cdot)$ is uniquely best. If $f \neq F(A, \cdot)$, $M(A) \cap V$ is empty and we apply the previous corollary. *Necessity.* We combine the nonuniqueness arguments of [5, Theorem 3] and the nonuniqueness arguments for generalized rational approximation to get a pair of approximations without zero-sign compatibility [3, 4].

In the classical case where w is the identity transformation $w(x, y) = y$ and we are approximating by R, the only fixed points of F are points x such that $P(\cdot, x) \equiv 0$, and $C(V, X)$ is the space of continuous functions vanishing on V. We obtain the

COROLLARY. *A necessary and sufficient condition that $R(A, \cdot)$ be uniquely best when it is best to $f \in C(V, X)$ is that $S(A)$ be a Haar subspace on $X \sim V$.*

REFERENCES

1. E. W. CHENEY, "Introduction to Approximation Theory," McGraw-Hill, New York, 1966
2. C. B. DUNHAM, Transformed rational Chebyshev approximation, *Numer. Math.* **10** (1967), 147–152.
3. C. B. DUNHAM, Chebyshev approximation by families with the betweeness property, *Trans. Amer. Math. Soc.* **136** (1969), 151–157.
4. C. B. DUNHAM, Chebyshev approximation with a null space, *Proc. Amer. Math. Soc.* **41** (1973), 557–558.
5. C. B. DUNHAM, Transformed linear Chebyshev approximation, *Aequationes Math.* **12** (1975), 6–11.
6. E. KAUFMAN AND G. BELFORD, Transformations of families of approximating functions, *J. Approximation Theory* **4** (1971), 363–371.
7. C. LAWSON, "Contributions to the theory of Linear Least Maximum Approximation," Dissertation, University of California, Los Angeles, 1961.
8. J. RICE, "The Approximation of Functions," Vol. 1, Addison-Wesley, Reading, Mass., 1964.
9. J. WILLIAMS, Numerical Chebyshev approximation by interpolating rationals, *Math. Comp.* **26** (1972), 199–206.